The Odds Meet the Great Martingale

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Outline

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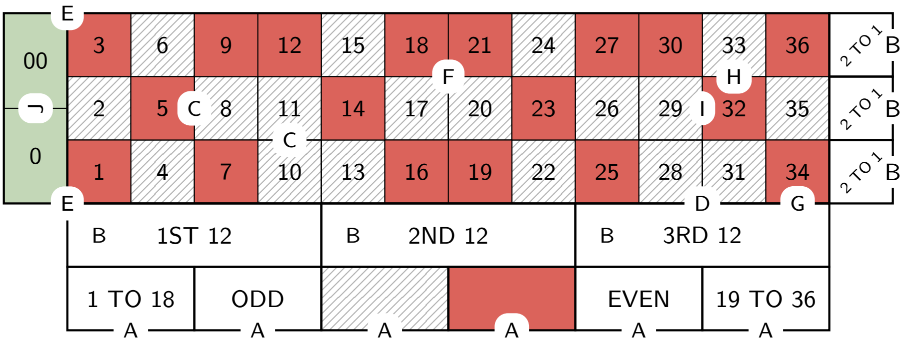
The American Roulette

- *A system to beat the odds in roulette? Yes, and here is a complete analysis of "the system"; the casinos' defense against "the system"; and your chances when restricted by the house limit.*
- One of the objectives of this article is to show just how well and under what conditions the Great Martingale system is able to stand up against the odds.

A Description of American Roulette

- There are thirty-eight numbers on the layout: 00, 0, 1, ..., 36.
  - 00 and 0 are green.
  - Eighteen numbers are red.
  - Eighteen numbers are black.
- Each number and color on the layout has a matching counterpart slot on the wheel.
- As the dealer starts the ball spinning around the wheel, the players place their bets on the layout.
- The different strategies with the corresponding payoffs are shown in figure 1

Layout



Strategy: Bet On	Payoff	*Strategy: Bet On*	Payoff
A-Red or Black, Odd or Even	Pays 1 to 1	F-17, 18, 20 and 21	Pays 8 to 1
B-1st 12, 2nd 12 or 3rd 12	Pays 2 to 1	G-34, 35 and 36	Pays 11 to 1
C-Any single number, 0 or 00	Pays 35 to 1	H-32 and 33	Pays 17 to 1
D-28, 29, 30, 31, 32 and 33	Pays 5 to 1	I-29 and 32	Pays 17 to 1
E-0, 00, 1, 2 and 3	Pays 6 to 1	J-0 and 00	Pays 17 to 1

Figure: The layout and payoff schedule

Outline

Payoffs

Ideal case

- The payoffs gives 0 expected profit if 0 and 00 doesn't exist.
- The strategy $S_n(p)$ with favorable scenarios $n < 36$ and payoff p per dollar must satisfies the following condition:

$$\mathrm{E}[S_n(p)] = \frac{n}{36} \times (1 + p) - 1 = 0 \iff p^* = \frac{36}{n} - 1$$

- For example, strategy D has 6 favorable must pay $p = \frac{36}{6} - 1 = 5$, and strategy G must pay $\frac{36}{3} - 1 = 11$.

Payoffs

Casino case

- The payoffs are calculated over 38 possibles cases, keeping the payoff p^* per dollar calculated in the ideal case.
- The expected payoff of any strategy will be:

$$\mathrm{E}(S_n(p^*)) = \frac{n}{38} \times \left[1 + \left(\frac{36}{n} - 1\right)\right] - 1 = -\frac{2}{38} \approx -5.26\%$$

- Each case (excepting one, however, with a greater expected loss) will result in a expected loss.
- So... there are a strategy???

Outline

Martingale System

- Suppose that income restriction isn't binding.
- If you lose your first bet, then double your bet plus one dollar, until you win.
- If the win occurs at the mth bet, you'll win m dollars.
- For simplicity, the strategy used for this system is A (betting on colors).

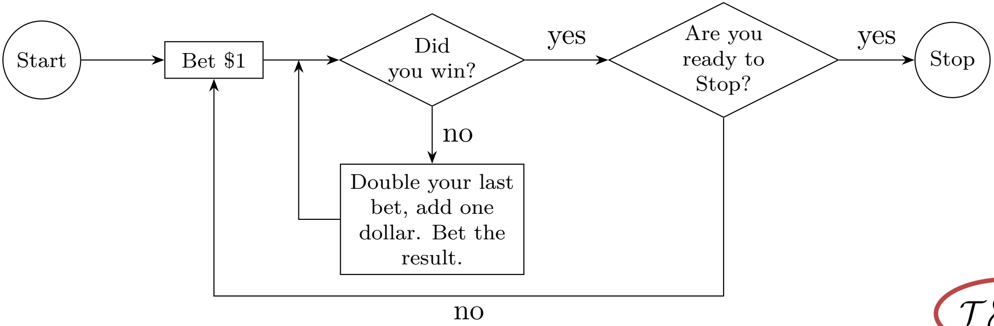


Figure: Martingale Flow Chart

Martingale System

Table: A simulated game using the martingale system

	Bets									
Description of Bet	1st	2d	3d	4th	5th	6th	7th	8th	9th	10th
Amount Bet	$1	$3	$1	$3	$7	$1	$3	$7	$15	$31
Outcome*	L	W	L	L	W	L	L	L	L	W
Profit on Bet	-$1	+$3	-$1	-$3	+$7	-$1	-$3	-$7	-$15	+$31
Accumulated Profit	-$1	+$2	+$1	-$2	+$5	+$4	+$1	-$6	-$21	+$10

*L represents a loss and W represents a win.

Martingale System

Theorem 1

If a player following the Great Martingale system terminates play with a win on the nth bet, the player will win exactly n dollars.

- *Proof:* Let k be the number of wins wins that occur during a game that ends with a win on the nth bet.
- Let $W_n(k)$ represent the player's profit at the end of n bets with k wins.
- Theorem 1 shows that, for all $k \in \{1, \ldots, n\}$, $W_n(k) = n$.

Martingale System

Proof of Theorem 1

- If $k = 1$ then the win occurs at the last bet. Martingale system impose that $b_1 = 1$, $b_2 = 3$, $b_3 = 7$, $b_4 = 15$ and so on:

$$b_{t+1} = 2b_t + 1 \implies b_t = 2^t - 1$$

$$\sum_{t=1}^{n-1} b_t = 2^n - 2 - (n-1) = 2^n - n - 1$$

- The final bet is equal to $b_n = 2^n - 1$. Then:

$$W_n(1) = 2^n - 1 - (2^n - n - 1) = n$$

Martingale System

Proof of Theorem 1

- Assume that $W_n(k) = n$ is true for $k > 1$. The proof will be completed if $W_n(k+1) = n$.
- If the kth win occurs at the rth bet, then the subgame involving r bets will give a payoff $W_r(r) = r$.
- The other subgame, from the starting bet $r+1$ to the last bet n only has one win, so $W_{n-r}(n-r) = n-r$. So, the total profit will be:

$$W_{n-r}(n-r) + W_r(r) = (n-r) + r = n \quad \square$$

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Outline

House limit

- Casinos impose a house limit of, commonly, \$500. The probability of win in at least one out of eight successive bets is:

$$1 - \left(\frac{20}{38}\right)^8 \approx 0.9941$$

- We take eight, because, if you lose eight bets then the bet must be $b_9 = 2^9 - 1 = \$511$ (the casino's restriction is binding).

House limit

Five Simulated Games	Number of Wins	Number of Losses	Percent Losses	Last Win	Last Bet	Total Game Profit
First	105	123	53.95%	220th	228th	−$282
Second	340	350	50.72%	682d	690th	+$180
Third	32	49	60.49%	73d	81st	−$429
Fourth	28	42	60.00%	62d	70th	−$440
Fifth	379	415	52.28%	786th	794th	+$284
Total	884	979	52.55%			−$687

Table: The martingale system versus a house limit of $500

- Profit when the last eight bets results in a lose is:

$$\pi(n) = n - 510$$

Expected Payoff

- The value of E[S] with the house restriction is:

$$\mathrm{E}[S] = \lim_{n\to\infty} \sum_{i=1}^{n} (f_i)(n-510)$$

- If $n < 8$ then $f_n = 0$.
- If $8 \leq n \leq 15$ then $f_n = (1-p)p^8$.
- If $n \geq 16$ then:

$$f_n \approx \frac{(x-1)(19-10x)}{9(9-8x)} \times \frac{1}{x^{n+1}}$$

where $x = \min\left\{|x'| : 1 - x' + \left(\frac{9}{19}\right)\left(\frac{10}{19}\right)^8 (x')^9 = 0\right\}$

- Using black magic:

$$\mathrm{E}[S] \approx -\$150$$

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Conclusion

- If you find a casino without a house limit, go and gamble!!!