# Cautious Expected Utility

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# First Things First: Preliminaries

**Lottery.** A lottery is a probability measure over outcomes in

$$X = [w, b] \subset \mathbb{R}.$$

That is, a lottery $p \in \Delta(X)$ satisfies

$$\int_X dp(x) = 1.$$

In the discrete case,

$$p = \{(x_i, \pi_i)\}_{i=1}^n, \qquad \sum_{i=1}^n \pi_i = 1, \qquad \pi_i \geq 0.$$

**Binary Relation.** A binary relation is defined as

$$\succsim \subseteq \Delta(X) \times \Delta(X).$$

For two lotteries $p, q \in \Delta(X)$. This means that, if $(p, q) \in \succsim$, then $p \succsim q$.

# Motivation: Allais' Paradox

Consider the following two choice problems:

| Problem | Lottery | Payoffs | |
|---|---|---|---|
| 1 | $A$ | 3000 w. probability 1 | |
| | $B$ | 4000 w. probability 0.8, | 0 w. probability 0.2 |
| 2 | $C$ | 3000 w. probability 0.25, | 0 w. probability 0.75 |
| | $D$ | 4000 w. probability 0.2, | 0 w. probability 0.8 |

The typical choice pattern is:

$$A \succ B \qquad \text{and} \qquad D \succ C.$$

*It can be shown that this does not correspond to the behavior of a rational agent (in the standard fashion).*

# Motivation: Allais' Paradox (cont.)

- The usual pattern shown before can be interpreted as follows: people tend to choose the safe option when certainty is available, which suggests that certainty itself has an additional value that standard expected utility does not capture well.
- In the remaining part of this presentation, we will construct a model that addresses this phenomenon while preserving most of the (axiomatic) structure behind standard expected utility maximization behavior.
- We will also show that the representation we obtain is unique in a well defined manner, and that a behaviorally equivalent one can be obtained through a specific completion of an incomplete preference relation over lotteries.

# Setup

**Prizes and Lotteries**

Let $[w, b] \subset \mathbb{R}$ be a compact interval of monetary prizes, and let $\Delta$ be the set of lotteries over $[w, b]$ (endowed with the topology of weak convergence).

We denote by $x, y, z$ generic prizes in $[w, b]$, and by $p, q, r$ generic lotteries in $\Delta$.

For each $x \in [w, b]$, let $\delta_x \in \Delta$ denote the degenerate lottery that gives prize $x$ with certainty.

The primitive of our analysis is a binary (preference) relation $\succsim$, where for two lotteries $p, q \in \Delta$, $p \succsim q$ means that the decision maker considers $p$ at least as good as $q$.

The certainty equivalent of a lottery $p \in \Delta$ is a prize

$$x_p \in [w, b] \quad \text{such that} \quad \delta_{x_p} \sim p.$$

# Basic Axioms

**Axiom 1 — Weak Order:** The relation $\succsim$ is complete and transitive.

This is the most basic notion of rationality: the decision maker can compare any two lotteries and does so consistently.

**Axiom 2 — Continuity:** For each $q \in \Delta$, the sets

$$\{p \in \Delta : p \succsim q\} \qquad \text{and} \qquad \{p \in \Delta : q \succsim p\}$$

are closed.

This is essential for representing preferences through continuous utility functions.

**Axiom 3 — Weak Monotonicity:** For each $x, y \in [w, b]$,

$$x \geq y \quad \Longleftrightarrow \quad \delta_x \succsim \delta_y.$$

This simply states that more money is at least as good as less money.

# Negative Certainty Independence (Key Axiom)

**Standard Independence**

For each $p, q, r \in \Delta$ and $\lambda \in (0, 1)$,

$$p \succsim q \quad \Longleftrightarrow \quad \lambda p + (1-\lambda) r \succsim \lambda q + (1-\lambda) r.$$

This means that if $p$ is preferred to $q$, then mixing both with the same third lottery $r$ should not reverse that ranking.

**Axiom 4 — Negative Certainty Independence (NCI)**

For each $p, q \in \Delta$, $x \in [w, b]$, and $\lambda \in [0, 1]$,

$$p \succsim \delta_x \quad \Longrightarrow \quad \lambda p + (1-\lambda) q \succsim \lambda \delta_x + (1-\lambda) q.$$

NCI says that if the sure outcome $x$ is not enough to compensate for the risky lottery $p$, then once certainty is removed by mixing both options with the same lottery $q$, the ranking should not reverse.

# Cautious Expected Utility Representation

**Utility Representation**

A function $V : \Delta \to \mathbb{R}$ represents $\succsim$ if

$$p \succsim q \quad \iff \quad V(p) \geq V(q).$$

Let $\mathcal{U}$ be the set of continuous and strictly increasing functions $u : [w, b] \to \mathbb{R}$ (endowed with the topology induced by the supnorm).

For each lottery $p$ and $u \in \mathcal{U}$, $E_p(u)$ denotes expected utility, and the certainty equivalent is given by

$$c(p, u) = u^{-1}\big(E_p(u)\big).$$

**Definition — Cautious Expected Utility**

A set $\mathcal{W} \subseteq \mathcal{U}$ is a cautious expected utility representation of $\succsim$ if the function $V : \Delta \to \mathbb{R}$

$$V(p) = \inf_{u \in \mathcal{W}} c(p, u) \qquad \forall p \in \Delta,$$

represents $\succsim$. We say $\mathcal{W}$ is continuous if $V$ is also continuous.

# Main Representation Theorem

**Theorem 1**

Let $\succsim$ be a binary relation on $\Delta$. The following statements are equivalent:

1. The relation $\succsim$ satisfies Weak Order, Continuity, Weak Monotonicity, and Negative Certainty Independence.
2. There exists a continuous cautious expected utility representation of $\succsim$.

This result shows that NCI is exactly the additional condition needed to move from standard expected utility to a cautious representation, which we will be capable of capturing the certainty effect.

# Interpretation

- Under cautious expected utility, the decision maker does not evaluate lotteries using a single utility function, but a set $\mathcal{W}$ of plausible utility functions.
- All functions in $\mathcal{W}$ agree that more money is better, but they may differ in curvature, reflecting different attitudes toward risk.
- When evaluating a risky lottery $p$, the agent behaves cautiously and uses the utility function that delivers the **lowest certainty equivalent**:

$$V(p) = \inf_{u \in \mathcal{W}} c(p, u).$$

- For degenerate lotteries, this caution disappears since

$$c(\delta_x, u) = x \qquad \forall u \in \mathcal{W}.$$

- This creates a natural advantage for sure outcomes and generates the **certainty effect**.

# Intuitive Example

Suppose a decision maker (DM) needs to evaluate a lottery $p$ that pays either \$0 or \$10,000 with equal probability. The DM might find it difficult to give a precise answer, but believes that a plausible valuation lies in the range $[\$3{,}500, \$4{,}500]$. By being cautious, she evaluates the lottery using the lowest value in that range and is willing to pay at most \$3,500.

# Formal Example

Let $\mathcal{W} = \{u_1, u_2\}$, where

$$u_1(x) = -\exp(-\beta x), \quad \beta > 0, \qquad u_2(x) = x^\alpha, \quad \alpha \in (0,1).$$

For $\alpha = 0.8$ and $\beta = 0.0002$,

$$V(B) = c(B, u_1) \simeq 2904 \ < \ 3000 = V(A),$$

while

$$V(D) = c(D, u_2) \simeq 535 \ > \ 530 \simeq c(C, u_2) = V(C).$$

Thus,

$$A \succ B \qquad \text{and} \qquad D \succ C,$$

replicating the Allais paradox.
If one utility function always delivered the lowest certainty equivalent, the model would collapse to standard expected utility.

**Normalized Utility Functions**

We define the set of normalized utilities as
$\mathcal{U}_{\text{nor}} = \{u \in \mathcal{U} : \ u(w) = 0, \ u(b) = 1\}$

Without loss of generality, we restrict attention to normalized representations, that is, $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$.

With this normalization, uniqueness of our representation will only be obtained *up to* the *closed convex hull*, which we will denote by $\overline{co}(\mathcal{W})$.

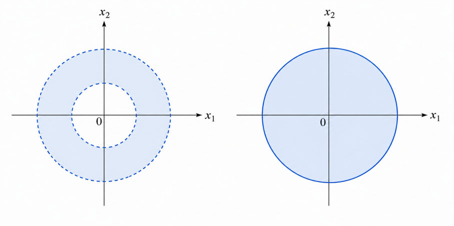


Figure: Intuitive visualization of $\mathcal{W}$ and $\overline{co}(\mathcal{W})$

# Uniqueness up to the Closed Convex Hull

**Proposition 1**

If

$$\overline{\text{co}}(\mathcal{W}) = \overline{\text{co}}(\mathcal{W}'),$$

then

$$\inf_{u \in \mathcal{W}} c(p, u) = \inf_{u \in \mathcal{W}'} c(p, u) \qquad \forall p \in \Delta.$$

This means that different sets of utility functions may generate exactly the same preference relation. Moreover, we can add redundant utility functions that never attain the infimum without changing the representation.

After removing these redundant elements, uniqueness is recovered only *up to the closed convex hull*.

With that being said, we will show that it is possible to identify a minimal set of relevant utilities that fully determines behavior.

# Minimal Set of Utilities

## Theorem 2

Let $\succsim$ be a binary relation on $\Delta$ satisfying Weak Order, Continuity, Weak Monotonicity, and Negative Certainty Independence.
Then there exists

$$\widehat{\mathcal{W}} \subseteq \mathcal{U}_{\text{nor}}$$

such that:

1. $\widehat{\mathcal{W}}$ is a continuous cautious expected utility representation of $\succsim$.
2. If $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$ is any other cautious expected utility representation of $\succsim$, then
$$\overline{\text{co}}(\widehat{\mathcal{W}}) \subseteq \overline{\text{co}}(\mathcal{W}).$$

Thus, $\widehat{\mathcal{W}}$ is the minimal set of utilities needed to represent behavior. From here on, if $\mathcal{W}$ is a cautious expected utility representation of a preference relation $\succsim$, we denote by $\widehat{\mathcal{W}}$ a set of utilities as described above.

# Characterization of Risk Attitudes

**Definition — Risk Attitude**

We say that $\succsim$ is

- **risk averse** if $p \succsim q$ whenever $q$ is a mean-preserving spread of $p$,
- **risk seeking** if $q \succsim p$ whenever $q$ is a mean-preserving spread of $p$.

**Theorem 3**

Let $\succsim$ satisfy Weak Order, Continuity, Weak Monotonicity, and Negative Certainty Independence. Then:

1. $\succsim$ is risk averse if and only if each $u \in \widehat{\mathcal{W}}$ is concave.
2. $\succsim$ is risk seeking if and only if each $u \in \widehat{\mathcal{W}}$ is convex.

Thus, the usual link between curvature and risk attitudes from expected utility still holds, but now for the entire set of relevant utilities. Note the DM could be risk seeking even despite the presence of the certainty effect.

# Comparative Risk Aversion

Let $\succsim_1$ and $\succsim_2$ be two preference relations on $\Delta$.
We say that $\succsim_1$ is **more risk averse** than $\succsim_2$ if for every $p \in \Delta$ and every $x \in [w, b]$, $p \succsim_1 \delta_x \quad \Longrightarrow \quad p \succsim_2 \delta_x$.

That is, whenever DM1 accepts a risky lottery over a sure amount, DM2 does so as well.

## Theorem 4

Let $\succsim_1$ and $\succsim_2$ be two binary relations with continuous cautious expected utility representations $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively.
The following statements are equivalent:

1. $\succsim_1$ is more risk averse than $\succsim_2$.
2. Both $\mathcal{W}_1 \cup \mathcal{W}_2$ and $\mathcal{W}_1$ are continuous cautious expected utility representations of $\succsim_1$.
3. $\overline{\text{co}}\left(\widehat{\mathcal{W}_1 \cup \mathcal{W}_2}\right) = \overline{\text{co}}\left(\widehat{\mathcal{W}_1}\right)$

# Interpretation of Comparative Risk Aversion

- DM1 is more risk averse than DM2 if adding the utilities of DM2 to the set of utilities of DM1 does not change the representation.
- In that case, all utilities in $\widehat{\mathcal{W}}_2$ are redundant once $\widehat{\mathcal{W}}_1$ is already considered.
- Intuitively, DM1 already evaluates lotteries at least as cautiously as DM2, and there are two channels that explain this:
  - the utilities in $\widehat{\mathcal{W}}_1$ are more concave (stronger local risk aversion),
  - or the set $\widehat{\mathcal{W}}_1$ is simply larger (more sources of caution).
- Thus, comparative risk aversion is influenced both by curvature and by the size of the relevant set of utilities.

# Linear Core and Indecisiveness: Separating Channels

**Linear Core**

Let $\succsim'$ be the largest subrelation of $\succsim$ that satisfies the independence axiom.

Formally, for $p, q \in \Delta$,

$$p \succsim' q \iff \lambda p + (1-\lambda) r \succsim \lambda q + (1-\lambda) r$$

for every $\lambda \in (0, 1]$ and every $r \in \Delta$.

The relation $\succsim'$ captures the comparisons that remain robust after mixing with any third lottery. Put differently, these are the comparisons the DM makes with confidence.

**Definition — More Indecisive**

We say that $\succsim_1$ is more indecisive than $\succsim_2$ if

$$p \succsim_1' q \implies p \succsim_2' q \qquad \forall p, q \in \Delta.$$

That is, whenever DM1 can confidently rank $p$ above $q$, DM2 can do so as well.

# Indecisiveness and Comparative Risk Aversion

## Proposition 2

Let $\succsim_1$ and $\succsim_2$ be two binary relations satisfying Weak Order, Continuity, Weak Monotonicity, and Negative Certainty Independence.
The following statements hold:

1. $\succsim_1$ is more indecisive than $\succsim_2$ if and only if

$$\overline{\text{co}}(\widehat{\mathcal{W}_2}) \subseteq \overline{\text{co}}(\widehat{\mathcal{W}_1}).$$

2. If $\succsim_1$ is more indecisive than $\succsim_2$, then $\succsim_1$ is more risk averse than $\succsim_2$.

Greater indecisiveness means fewer confident comparisons, which translates into more cautious behavior and therefore greater risk aversion.

# Cautious Completions of Incomplete Preferences

- So far, we studied complete preference relations satisfying NCI.
- The paper now considers a DM with an **incomplete** preference relation over lotteries: some pairs of lotteries may not be comparable.
- Suppose the DM is forced to choose and completes preferences by applying caution: when in doubt between a sure outcome and a lottery, she chooses the sure outcome.
- The question is: what complete preference relation emerges from this cautious completion?

**Axiom 5 — Sequential Continuity**

$$p_n \to p, \quad q_n \to q, \quad p_n \succsim q_n \ \forall n \quad \Longrightarrow \quad p \succsim q.$$

This is a stronger continuity requirement, needed because preferences are now incomplete.

**Definition — Cautious Completion**

Let $\succsim'$ be an incomplete binary relation on $\Delta$.
We say that a binary relation $\hat{\succsim}$ is a **cautious completion** of $\succsim'$ if:

1. $\hat{\succsim}$ satisfies Weak Order, Weak Monotonicity, and for each $p \in \Delta$, there exists $x_p \in [w, b]$ such that $p \,\hat{\sim}\, \delta_{x_p}$
2. For every $p, q \in \Delta$, if $p \succsim' q$, then $p \,\hat{\succsim}\, q$.
3. For every $p \in \Delta$ and $x \in [w, b]$, if $p \not\succsim' \delta_x$, then $\delta_x \,\hat{\succ}\, p$.

The idea is simple: when the original preference relation cannot compare a sure outcome and a lottery, the completion chooses the sure outcome.

# Cautious Completion

## Theorem 5

If $\succsim'$ is a reflexive and transitive binary relation on $\Delta$ satisfying Sequential Continuity, Weak Monotonicity, and Independence, then:

1. $\succsim'$ admits a unique cautious completion $\hat{\succsim}$
2. There exists a set $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$ such that for all $p, q \in \Delta$,

$$p \succsim' q \iff E_p(u) \geq E_q(u) \qquad \forall u \in \mathcal{W},$$

and

$$p \mathrel{\hat{\succsim}} q \iff \inf_{u \in \mathcal{W}} c(p, u) \geq \inf_{u \in \mathcal{W}} c(q, u).$$

Moreover, $\mathcal{W}$ is unique up to the closed convex hull.

# The end